

## 34 IMPORTANT THINGS TO KNOW BEFORE JANUARY 7

### *Mechanics.*

1. You will frequently have to decide whether to use Newton's laws or canonical (Lagrangian or Hamiltonian) methods to solve a problem. If the system involves a nontrivial holonomic constraint (most commonly a particle is constrained to a curve, frequently a rotating one), or if the natural coordinate system is rotating, it is very desirable to use canonical methods when possible because it is easy to forget important terms in the Newtonian method. In some cases the system is nonconservative and no Lagrangian exists, in which case it may be preferable to "patch" frictional forces onto your resulting equations of motion rather than solve the problem by Newtonian means. (The Rayleigh function is a systematic way of doing this but none of the problems we've done have required anything that sophisticated.)

2. Conservation laws are key to many problems. Remember that a conservation law applies if its associated symmetry is valid. A conservative system with no explicit time dependence in the Hamiltonian will have conserved Hamiltonian. (Sometimes this may be different from the total energy.) Spatial homogeneity is associated with conservation of canonical linear momentum and isotropy with conservation of canonical angular momentum. If in *any* doubt about the conservation law associated with a certain symmetry, write out the Lagrangian in a coordinate system that takes advantage of the symmetry and directly find the cyclic conjugate momenta (if  $q_k$  is not explicitly in the Lagrangian, then  $p_k = \partial L / \partial \dot{q}_k$  is conserved).

3. Understand your frames of reference; sometimes you will need to work in more than one frame of reference to arrive at an answer. When working with rotating systems, especially in 3 dimensions, be certain to understand your choice of axes (e.g. which are body-fixed vs. inertial). The time derivative of a vector quantity is different in the inertial vs. rotating frame. Dynamical equations of motion based on torque equalling the time-derivative of angular momentum are always constructed in the inertial frame. The conversion can be obtained by applying the product rule to each component, e.g. if  $x', y', z'$  are body-fixed coordinates rotating at inertial angular velocity  $\omega$  and  $x, y, z$  are inertial, then for vector  $\mathbf{A}$ :

$$(\dot{\mathbf{A}})_{\text{body}} = (\dot{\mathbf{A}})_{\text{inertial}} + \mathbf{A} \times \omega$$

Also be careful with accelerating reference frames, and with inertial reference frames moving at different velocities (e.g. center of mass vs. lab frame).

4. Not all small oscillations are simple harmonic oscillations. Make sure you understand the shape of the potential. As a sanity check, if the potential increases with displacement faster than  $x^2$ , higher-amplitude oscillations will

have shorter periods, whereas if the potential increases with displacement slower than  $x^2$ , the reverse is true. For a general potential in one dimension, the fastest way to determine the period of oscillations at arbitrary energy is usually to use the method of angle-action variables:  $\tau = \partial J / \partial H$ , where  $H$  is the (conserved!) Hamiltonian, and  $J$  is the action integral:

$$J = \oint pdq = 2 \int_{x_{\min}}^{x_{\max}} \sqrt{2m(E - V)} dx$$

where the last equality holds for the case of a simple potential.

5. Central-force problems are (almost) invariably solved by working in polar coordinates, and then using conservation of angular momentum to eliminate the angular variable. The major exception is that  $F \propto r$  central-force problems are simple harmonic oscillators. For force laws that fall off steeply ( $F \propto 1/r^3$  and steeper), radial oscillations are not stable. Several problems have asked about the relation between the radial oscillation frequency and the angular motion frequency.

6. When using conservation of energy to solve a problem, always make sure to include all forms, sources, and sinks of energy. This includes translation, rotation, elasticity, gravitation, etc. The same is true for other conservation laws; tricky question writers will hide other forms of energy or momentum in the problem. External power, force, and torque result in contributions to the system's energy, linear momentum, and angular momentum, respectively. Normal contact forces on a stationary surface do no work. Static friction does no work. Kinetic friction does negative work. You can tell if a force produces a torque around a given point by looking at its line of action.

7. When working with a continuous medium (e.g. a rope), it is usually a good idea to actually write down Newton's laws for a small element of the medium, or write down the kinetic and potential energies and construct a Lagrangian. Most problems where the rope plays a central role will have varying loads along the rope in such a way that no simpler approach exists, and to attempt one risks serious error.

### ***Electricity and Magnetism.***

8. The Laplace/Poisson equation can be solved using image charges for planar mirrors in 3D and linear and circular mirrors in 2D; other situations in which image charges are useful are rare. Orthogonal-function methods tend to be useful if you suspect the "answer" to the problem is supposed to be in terms of an infinite series; if all but a few of the terms in the expansion vanish; or the problem asks for the total energy of the field, in which case the contributions from the different modes usually "just add up." In some problems, using orthogonal functions in only one of the coordinates is useful, e.g. if an object is placed in a uniform background (external dipole) field plus an external quadrupole field, only the  $l = 1$  and  $l = 2$  modes are excited, and so it may

make sense to take a potential of the form  $\Phi = A(r)Y_1 + B(r)Y_2$  with differential equations for  $A$  and  $B$ , where  $Y_1$  and  $Y_2$  are linear combinations of spherical harmonics appropriate to the boundary conditions.

9. In 2D, conformal transformations can be of use; in constructing these, note that the exponential function maps a Cartesian grid to a polar-coordinate grid, and vice versa for the logarithm. Power functions  $z \mapsto z^n$  will keep the polar coordinate grid but expand angles by a factor of  $n$ . Also the Mobius function,  $z \mapsto (az + b)/(cz + d)$  with  $ad - bc \neq 0$  can map any line or circle to any other line or circle for some choice of coefficients. Finally, any polygonal region in the  $w$ -plane can be mapped onto the upper half complex  $z$ -plane by the Schwarz-Christoffel transformation:

$$w = A \int dz \prod_{k=1}^n (z - x_k)^{\phi_k/\pi - 1}$$

where the  $\phi_k$  are the angles in the polygon.

10. When an electromagnetic or plasma wave is propagating in matter, a simple and reliable way to find the dispersion relation and allowed polarization is to take Maxwell's equations and the macroscopic equations relating  $\mathbf{E}$  to  $\mathbf{D}$  and  $\mathbf{H}$  to  $\mathbf{B}$  and substitute  $\nabla \mapsto i\mathbf{k}$ ,  $\partial_t \mapsto -i\omega$ . If there are other degrees of freedom, such as displacement  $\mathbf{x}$  of electrons in a plasma, these can be grafted onto the method.

11. Radiation problems are most easily solved either by the Larmor formula when applicable. Otherwise, the most straightforward method is to use the retarded Green's function in Lorentz gauge. Sometimes radiation and image-charge results must be combined; in these cases, you can use the appropriate surface boundary conditions to determine the image charge and current distribution, find the radiation field from the Green's function, and (if necessary) integrate over the open  $2\pi$  solid angle. Occasionally, Larmor's formula combined with a simple interference argument will solve a problem.

12. Be familiar with the solution to simple electrostatic and magnetostatic problems. The equipotentials and field lines of two straight, infinitely long, parallel and equal/oppositely charged wires are cylinders and circles, respectively (except in the degenerate case of a plane or a line.)

13. Try to replace vector equations with equations for scalar potentials. In static situations,  $\mathbf{E} = -\nabla\Phi$  holds, even if dielectrics are present (in which case Laplace's equation is modified.) Similarly, in the absence of free currents or displacement currents,  $\mathbf{H} = \nabla\Gamma$  holds for some function  $\Gamma$ , even if magnetic materials are present.

### **Quantum Mechanics.**

14. The key to almost every time-dependent perturbation theory problem is Fermi's Golden rule:

$$\Gamma = \frac{2\pi}{\hbar^2} w(E) | \langle i | \Delta H | f \rangle |^2$$

where  $\Delta H$  is the resonant piece of the perturbation Hamiltonian,  $|i\rangle$  and  $|f\rangle$  are the initial and final states,  $w(E)$  is the density of states, and  $\Gamma$  is the transition rate. There are  $h^{-n}$  states per unit volume in  $xp$ -space, where  $n$  is the number of dimensions. The volume of your box thus appears in the density of states, and will cancel against the normalization of the states.

15. Spontaneous emission from a system can be handled with Einstein's equation:

$$A = \frac{8\pi h\nu^3}{c^3} B$$

where  $A$  is the spontaneous emission probability (e.g. in  $s^{-1}$ ), and  $B$  is the rate of stimulated emission divided by electromagnetic energy density per unit frequency at the transition frequency. (This relation derives from consideration of thermal equilibrium.) This is useful because  $B$  can be computed from Fermi's Golden rule.

16. The Born approximation formula is:

$$\frac{d\sigma}{d\Omega} = \left( \frac{m}{2\pi\hbar^2} \right)^2 \left| \int_{\mathbf{R}^3} V(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}} d^3\mathbf{r} \right|^2$$

where  $\mathbf{q}$  is the momentum transfer (final minus initial momentum divided by  $\hbar$ .) It can be extended to 2D scattering problems by introducing a passive third dimension (taking the 2D potential and making it very long in  $z$ .)

17. Many problems that look like time-independent perturbation theory can be solved exactly with very little effort. This is the case typically for problems with simple harmonic oscillators, a magnetic field and a linear or quadratic (in  $x$ ) perturbation. Otherwise, a state  $|n\rangle$  will have perturbed energy (to second order):

$$E'_n = E_n + \langle n | W | n \rangle + \sum_{m \neq n} \frac{|\langle m | W | n \rangle|^2}{E_n - E_m}$$

and the perturbed state is (to first order):

$$|n'\rangle = |n\rangle + \sum_{m \neq n} \frac{\langle m | W | n \rangle}{E_n - E_m} |m\rangle$$

18. Exploiting symmetry is the key to finding good quantum numbers. Determine whether the Hamiltonian commutes with a certain operator. You will also need to check that your operators associated with each quantum number commute with each other. Try to group spins together to create spin quantum numbers that commute with the spatial portion of the Hamiltonian (if any).

19. If you see a nasty-looking hydrogen molecule problem, avoid it.

***Thermodynamics and Statistical Mechanics.***

20. Most of the "solving a system"-type problems will require the use of formulas for intensive and extensive variables. These are related through the thermodynamic identity:

$$\frac{dS}{k_B} = \sum_{j=1}^n \beta I_j dX_j$$

where  $I_j$  are intensive variables and  $X_j$  are extensive variables. What holds for one intensive-extensive pair holds for them all. Examples of intensive-extensive pairs  $(I_j, X_j)$  are pressure-volume  $(P, V)$ , chemical potential-particle number  $(-\mu, N)$ , surface tension-area  $(-\gamma, A)$ , linear tension-length  $(-F, L)$ , magnetic field-magnetic moment  $(-H, M)$ , and one-energy  $(1, E)$ .

21. The partition function for an ensemble can be written as:

$$Z(\{I_j\}_{j \in J}, \{X_j\}_{j \notin J}) = \sum_{\text{states}} \exp\left(-\sum_{j \in J} \hat{I}_j X_j\right)$$

where  $\hat{I}_j = \beta I_j$ ,  $J$  is the set of intensive-extensive variables for which the *intensive* rather than the extensive variable is fixed, that is, the set of variables for which the extensive quantity can be exchanged with a reservoir. The contribution of a state to the sum is in direct proportion to its equilibrium probability. Examples include the microcanonical ensemble where  $J = \emptyset$ , the canonical ensemble  $J = \{E\}$ , and the grand canonical ensemble  $J = \{E, N\}$ . The partition function is a function of the fixed parameters.

22. The values of the extensive variables  $(\{X_j\}, j \in J)$  can be determined by differentiation:

$$X_j = -\frac{\partial}{\partial \hat{I}_j} \ln Z$$

where all other  $\{\hat{I}_i\}_{i \in J}$  are held constant. This is most commonly used in the case of the energy, where  $\hat{I}_E = \beta$ , and the number of particles, where  $\hat{I}_N = -\beta\mu$ . Note that for variables other than the energy,  $\beta$  can be considered a constant in the differentiation, so for these variables,  $X_j = -kT \partial_{I_j} \ln Z$ . For the non-fixed intensive variables,  $\{I_j\}_{j \notin J}$ , we have that:

$$I_j = \beta^{-1} \hat{I}_j = \beta^{-1} \frac{\partial}{\partial X_j} \ln Z$$

23. These parameters may alternatively be computed using a Legendre transformation. We define the enthalpy,  $E^* = \sum_{j \in J} I_j X_j$ , and free energy  $A = E^* - TS$ . Then we have that  $A = -\beta^{-1} \ln Z$ , and the non-fixed intensive

variables are  $\hat{I}_j = -\partial\hat{A}/\partial X_j$  (where, again, the hats represent multiplication by  $\beta$ .) The non-fixed extensive variables are  $X_j = \partial\hat{A}/\partial\hat{I}_j$ .

24. Statistical mechanics is famous for its large number of partial derivative relations. Don't be fooled. The relations in [22] and [23], and the partial derivative relations that follow trivially from the thermodynamic identity, combined with the less frequently used Maxwell relations (equality of mixed second derivatives of the free energy for various ensembles) cover all the partial derivative relations in elementary statistical mechanics.

25. Equilibrium is the state of maximum total entropy, hence for any allowed variation,  $\delta S = 0$ . The thermodynamic identity can be used to trivially identify the equations that must hold for this to be true.

26. For fermions, the mean occupation number at the chemical potential is  $1/2$ ; for bosons, the chemical potential is always below the ground state energy. At the very low temperatures, fermions become degenerate whereas bosons collect near the ground state. The chemical potential for fermions becomes the Fermi energy and for bosons becomes near the ground state energy. BEC forms when the total occupation number of excited states is less than the number of particles, even for chemical potential at the ground state energy.

27. In engine problems, it is usually easier to solve for heat input and output, and subtract to find work, rather than the reverse. *Always* draw the cycle in the  $TS$ -plane; this is helpful since heat transfer is  $\int TdS$ , and heat flows into the working fluid for  $dS > 0$ , out for  $dS < 0$ . Make sure you know which direction heat is flowing, and beware changes in direction partway through a stage.

### **General.**

28. The basic trigonometric identities (Pythagorean rules, sum rules, double-angle and half-angle rules, sum-to-product and product-to-sum rules) are essential to obtaining simple answers to many problems. Also be familiar with various integration tricks, such as clever substitutions, completing the square in Gaussian integrals, gamma integrals, integration by parts, and partial fractions.

29. Even in problems with all real variables, complex numbers can be a time-saving tool. Two examples which have appeared in past exams are the integration of exponential-times-trigonometric integrands, e.g.:

$$\int_0^\infty x^n e^{-\alpha x} \sin(\beta x + \phi) dx = \Im e^{i\phi} \int_0^\infty x^n e^{-(\alpha - i\beta)x} dx = \Im \frac{e^{i\phi} \Gamma(n+1)}{(\alpha - i\beta)^{n+1}}$$

and in the solution of equation of motion in a system with a  $\Xi \hat{\mathbf{z}} \times \mathbf{v}$  force (e.g. Coriolis force, constant magnetic field on charged particle). In such cases, it is sometimes (although definitely not always) useful to solve the equations of motion using the substitution  $w = x + iy$ , thus substantially reducing your workload.

30. In the solution of many problems involving differential equations (e.g. waves on a string, TEM pulse reflection on a coaxial cable, quantum scattering from a square well), very careful attention must be paid to the "boundary" conditions, particularly at the transition points.

31. In some problems, there are several parameters going to zero (or infinity) simultaneously, frequently with relations between them, e.g.  $k \rightarrow 0$ ,  $L \rightarrow \infty$ , but  $kL = N = \text{const}$ . In such cases, it is best to substitute out one of the variables (here  $k$  or  $L$ ) in favor of a constant ( $N$ ) before taking any limits. Taking  $k$  to zero first and then  $L$  to infinity may give the wrong answer!

32. When using a Taylor expansion, be sure to keep *all* terms to the relevant order. There may be several "lowest-order" contributions to the effect you are trying to evaluate, all of the same order.

33. Step back and take a look at the problem, before and after you solve it. If it mentions oscillations in  $\phi$ , think about why oscillations in  $\phi$  would occur. Think about scaling laws that you would expect your answer to obey, symmetries, and what sign, units, and limiting behavior it should have. Draw a diagram if necessary (e.g. graph a given potential). This may help you solve the problem, and it will catch most errors. *Always* pay attention to red flags. If something looks wrong, maybe it is.

34. Do not be afraid to apply mathematical methods that you normally think of as being "another part of physics." For example, on previous exams, multipoles and image solution to differential equations and (some variant of) Bloch's theorem have appeared in the mechanics section.